

# Resolvable $G$ -designs of order $v$ and index $\lambda$

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## Abstract

In this paper we consider the problem concerning the existence of a resolvable  $G$ -design of order  $v$  and index  $\lambda$ . We solve the problem for the cases in which  $G$  is a connected subgraph of  $K_4$ .

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## 1 Introduction and definitions

Let  $v$  and  $\lambda$  be positive integers,  $\lambda K_v$  be the complete multigraph of order  $v$  and index  $\lambda$  and  $G$  be a subgraph of  $K_v$ . A  $G$ -design of order  $v$  and index  $\lambda$  (denoted by  $(\lambda K_v, G)$ -design), is a decomposition of the edge set of  $\lambda K_v$  into subgraphs (called *blocks*) isomorphic to  $G$ . A  $(\lambda K_v, G)$ -design is said to be *resolvable* if it is possible to partition the blocks into classes  $\mathcal{P}_i$  (often referred to as *parallel classes*) such that every vertex of  $\lambda K_v$  appears in exactly one block of each  $\mathcal{P}_i$ . By simple calculation, we can obtain the following result.

**Lemma 1.1.** *If there exists a resolvable  $(\lambda K_v, G)$ -design, then  $v \equiv 0 \pmod{|V(G)|}$ ,  $\lambda v(v-1) \equiv 0 \pmod{2|E(G)|}$ , and  $\lambda(v-1) \equiv 0 \pmod{2|E(G)|}$ .*

The Kirkman schoolgirl problem has developed the following question in the theory of resolvable  $G$ -designs: “For a fixed graph  $G$  and a index  $\lambda$ , what are necessary and sufficient conditions for the existence of a resolvable  $(\lambda K_v, G)$ -design?” The Kirkman schoolgirl problem is this question for  $G = K_3$  and  $\lambda = 1$ , posed by Kirkman ([9]) in 1847 and solved by Hanani, Ray-Chaudhuri and Wilson ([7]) in 1969. The question has been studied for:  $G = K_4$  and  $\lambda = 1, 3$  by Hanani, Ray-Chaudhuri and Wilson ([7]);  $G = K_3$  and  $\lambda = 2$  by Hanani ([6]);  $G = P_3$  and every admissible  $\lambda$  by Horton ([8]);  $G = P_k, k \geq 4$  and every admissible  $\lambda$  by Bermond, Heinrich and Yu ([1]);  $G = K_4 - e$  and  $\lambda = 1$  by Ge, Ling, Colbourn, Stinson, Whang and Zhu ([3, 5, 14]).

The existence of a resolvable decomposition when  $G$  is a subgraph of  $K_4$  was studied separately already long ago:

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- There exists a resolvable  $(\lambda K_v, K_2)$ -design if and only if  $v \equiv 0 \pmod{2}$  and  $\lambda \geq 1$ .
- There exists a resolvable  $(\lambda K_v, P_3)$ -design if and only if  $v \equiv 0 \pmod{3}$  and  $\lambda(v-1) \equiv 0 \pmod{4}$  ([8]).
- There exists a resolvable  $(\lambda K_v, K_3)$ -design if and only if  $\lambda \equiv 0 \pmod{2}$  for  $v \equiv 0 \pmod{3}$ ,  $v \neq 6$ , or  $\lambda \geq 1$  for  $v \equiv 3 \pmod{6}$  ([6, 7]).
- There exists a resolvable  $(\lambda K_v, P_4)$ -design if and only if  $v \equiv 0 \pmod{4}$  and  $4\lambda(v-1) \equiv 0 \pmod{6}$  ([1]).
- There exists a resolvable  $(K_v, K_4 - e)$ -design if and only if  $v \equiv 16 \pmod{20}$  or  $v \equiv 116 \pmod{120}$  ([3, 5, 15]).
- There exists a resolvable  $(\lambda K_v, K_4)$ -design if and only if  $\lambda \equiv 0 \pmod{3}$  for  $v \equiv 0, 8 \pmod{12}$  or  $\lambda \geq 1$  for  $v \equiv 4 \pmod{12}$  ([6, 7]).

In this paper we shall focus our attention on the problem of the existence of resolvable  $(\lambda K_v, G)$ -designs when  $G = C_4, K_3 + e, K_{1,3}, K_4 - e$ , solving the spectrum problem for any connected subgraph of  $K_4$ .

In what follows, we will denote:

- by  $[a_1, a_2, \dots, a_k]$  the path  $P_k$ ,  $k \geq 3$ , having vertex set  $\{a_1, a_2, \dots, a_k\}$  and edge set  $\{\{a_1, a_2\}, \{a_2, a_3\}, \dots, \{a_{k-1}, a_k\}\}$ ,
- by  $(a_1, a_2, a_3, a_4)$  the 4-cycle  $C_4$  having vertex set  $\{a_1, a_2, a_3, a_4\}$  and edge set  $\{\{a_1, a_2\}, \{a_2, a_3\}, \{a_3, a_4\}, \{a_4, a_1\}\}$ ,
- by  $(a_1; a_2, a_3, a_4)$  the 3-star  $K_{1,3}$  having vertex set  $\{a_1, a_2, a_3, a_4\}$  and edge set  $\{\{a_1, a_2\}, \{a_1, a_3\}, \{a_1, a_4\}\}$ ,
- by  $(a_1, a_2, a_3; a_4)$  the graph  $K_4 - e$  having vertex set  $\{a_1, a_2, a_3, a_4\}$  and edge set  $\{\{a_1, a_2\}, \{a_1, a_3\}, \{a_2, a_3\}, \{a_1, a_4\}, \{a_2, a_4\}\}$  and
- by  $(a_1, a_2, a_3 - a_4)$  the kite  $K_3 + e$  having vertex set  $\{a_1, a_2, a_3, a_4\}$  and edge set  $\{\{a_1, a_2\}, \{a_1, a_3\}, \{a_2, a_3\}, \{a_3, a_4\}\}$ .

## 2 Necessary conditions

In this section we will give necessary conditions for the existence of a resolvable  $(\lambda K_v, G)$ -design.

**Lemma 2.1.** *If there exists a resolvable  $(\lambda K_v, G)$ -design, with  $G \in \{C_4, K_3 + e\}$ , then  $v \equiv 0 \pmod{4}$  and  $\lambda \equiv 0 \pmod{2}$ .*

*Proof.* By Lemma 1.1

$$v \equiv 0 \pmod{4}, \quad \lambda v(v-1) \equiv 0 \pmod{8}, \quad \lambda(v-1) \equiv 0 \pmod{2},$$

and so the conclusion follows.  $\square$

**Lemma 2.2.** *If there exists a resolvable  $(\lambda K_v, K_{1,3})$ -design, then  $v \equiv 0 \pmod{4}$  and, in particular:*

- i) if  $v \equiv 4 \pmod{12}$ , then  $\lambda \equiv 0 \pmod{2}$ ;*
- i) if  $v \equiv 0, 8 \pmod{12}$ , then  $\lambda \equiv 0 \pmod{6}$ .*

*Proof.* By Lemma 1.1

$$v \equiv 0 \pmod{4}, \quad \lambda v(v-1) \equiv 0 \pmod{6}, \quad 2\lambda(v-1) \equiv 0 \pmod{3}.$$

Now proceeding as in the proof of Lemma 2.1 of [10] the number of the parallel classes must be  $\equiv 0 \pmod{4}$  and the conclusion follows.  $\square$

**Lemma 2.3.** *If there exists a resolvable  $(\lambda K_v, K_4 - e)$ -design, then  $v \equiv 0 \pmod{4}$  and, in particular:*

- i) if  $v \equiv 0, 4, 8, 12 \pmod{20}$ , then  $\lambda \equiv 0 \pmod{5}$ ;*
- i) if  $v \equiv 16 \pmod{20}$ , then  $\lambda$  is any positive integer.*

*Proof.* By Lemma 1.1

$$v \equiv 0 \pmod{4}, \quad \lambda v(v-1) \equiv 0 \pmod{10}, \quad 2\lambda(v-1) \equiv 0 \pmod{5},$$

which implies the thesis.  $\square$

### 3 Costructions and related structures

In this section we will introduce some useful definitions. For missing terms or results that are not explicitly explained in the paper, the reader is referred to [2] and its online updates. For some results below, we also cite this handbook instead of the original papers.

A (resolvable)  $G$ -decomposition of the complete multipartite graph with  $u$  parts each of size  $g$  is known as a (resolvable) group divisible design  $G$ -(R)GDD of type  $g^u$  (the parts of size  $g$  are called the *groups* of the design). When  $G = K_n$  we will call it an  $n$ -(R)GDD. If the blocks of a  $G$ -GDD of type  $g^u$  can be partitioned into partial parallel classes, each of which contains all points except those of one group, we refer to the decomposition as a *frame*. It is easy to deduce that the number of partial parallel classes missing a specified group is  $\frac{g|V(G)|}{2|E(G)|}$ .

An incomplete resolvable  $G$ -design of order  $v + h$  and index  $\lambda$  with a hole of size  $h$  denoted by  $G$ -IRD( $v + h, h, \lambda$ ), is a  $G$ -decomposition of  $\lambda(K_{v+h} \setminus K_h)$  in which there are two types of classes,  $\frac{\lambda(h-1)|V(G)|}{2|E(G)|}$  *partial* classes which cover every point except those in the hole (the set of points of  $K_h$  are referred to as the *hole*) and  $\frac{\lambda v|V(G)|}{2|E(G)|}$  *full* classes which cover every point of  $K_{v+h}$ .

### 4 Small cases

**Lemma 4.1.** *There exists a resolvable  $(2K_4, C_4)$ -design.*

*Proof.* Let  $V = \{0, 1, 2, 3\}$  be the vertex set and consider the classes listed below:  
 $\{(0, 1, 2, 3)\}, \{(0, 1, 3, 2)\}, \{(0, 2, 1, 3)\}.$  □

**Lemma 4.2.** *There exists a resolvable  $(2K_4, K_3 + e)$ -design.*

*Proof.* Let  $V = \{0, 1, 2, 3\}$  be the vertex set and consider the classes listed below:  
 $\{(0, 2, 3 - 1)\}, \{(3, 2, 1 - 0)\}, \{(2, 1, 0 - 3)\}.$  □

**Lemma 4.3.** *There exists a resolvable  $(2K_8, K_3 + e)$ -design.*

*Proof.* Let  $V = Z_7 \cup \{\infty\}$  be the vertex set. The desired design is obtained by developing in  $Z_7$  the following base blocks:  $\{(\infty, 1, 5 - 6), (0, 4, 2 - 3)\}.$  □

**Lemma 4.4.** *There exists a resolvable  $(2K_4, K_{1,3})$ -design.*

*Proof.* Let  $V = Z_4$  be the vertex set. The desired design is obtained by developing in  $Z_4$  the base block  $\{(0; 1, 2, 3)\}$ .  $\square$

**Lemma 4.5.** *There exists a resolvable  $K_{1,3}$ -RGDD of type  $4^2$  and index 6.*

*Proof.* Take  $\{x_1, x_2, x_3, x_4\}$  and  $\{y_1, y_2, y_3, y_4\}$  as groups and consider the classes obtained by developing the following base blocks, reducing subscripts modulo 4:  
 $\{(x_1; y_1, y_2, y_3), (y_4; x_2, x_3, x_4)\}$ ,  $\{(x_1; y_2, y_3, y_4), (y_1; x_2, x_3, x_4)\}$ ,  $\{(x_1; y_3, y_4, y_1), (y_2; x_2, x_3, x_4)\}$ ,  $\{(x_1; y_4, y_1, y_2), (y_3; x_2, x_3, x_4)\}$ .  $\square$

**Lemma 4.6.** *There exists a resolvable  $K_{1,3}$ -RGDD of type  $4^3$  and index 3.*

*Proof.* Take  $3Z_{12} + i$ ,  $i = 0, 1, 2$ , as groups and the blocks obtained by developing  $(2; 0, 7, 9)$ , which gives four partial parallel classes, and the base blocks  $(4; 0, 3, 8)$ ,  $(6; 1, 2, 5)$ ,  $(9; 7, 10, 11)$  (giving a full parallel classe).  $\square$

**Lemma 4.7.** *There exists a resolvable  $(6K_8, K_{1,3})$ -design.*

*Proof.* Start with a resolvable  $K_{1,3}$ -RGDD of type  $4^2$  and index 6, which exists by Lemma 4.5, and fill each group of size 4 with a copy of a resolvable  $(6K_4, K_{1,3})$ -design, which exists by Lemma 4.4.  $\square$

**Lemma 4.8.** *There exists a resolvable  $(6K_{12}, K_{1,3})$ -design.*

*Proof.* Start with a resolvable  $K_{1,3}$ -RGDD of type  $4^3$  and index 6, which exists by Lemma 4.6, and fill each group of size 4 with a copy of a resolvable  $(6K_4, K_{1,3})$ -design, which exists by Lemma 4.4.  $\square$

**Lemma 4.9.** *There exists a resolvable  $(6K_{20}, K_{1,3})$ -design.*

*Proof.* Let  $V = Z_{19} \cup \{\infty\}$  be the vertex set. The desired design is obtained by developing in  $Z_{19}$  the following base blocks:

$\{(\infty; 4, 5, 12), (0; 3, 9, 11), (8; 1, 2, 13), (10; 7, 14, 18), (15; 6, 16, 17)\}$ ,  
 $\{(11; \infty, 9, 12), (0; 3, 4, 5), (8; 1, 2, 13), (10; 7, 14, 18), (15; 6, 16, 17)\}$ ,  
 $\{(5; \infty, 3, 11), (0; 4, 9, 12), (8; 1, 2, 13), (10; 7, 14, 18), (15; 6, 16, 17)\}$ ,  
 $\{(3; \infty, 9, 4), (0; 5, 12, 11), (8; 1, 2, 13), (10; 7, 14, 18), (15; 6, 16, 17)\}$ .  $\square$

**Lemma 4.10.** *There exists a resolvable  $(6K_{24}, K_{1,3})$ -design.*

*Proof.* Start with a resolvable 2-RGDD of type  $1^6$ . Give weight 4 to all points and replace each edge of a given resolution class with a copy of a resolvable  $K_{1,3}$ -RGDD of type  $4^2$  and index 6, which exists by Lemma 4.5. Finally, fill each group of size 4 with a copy of a resolvable  $(6K_4, K_{1,3})$ -design, which exists by Lemma 4.4.  $\square$

**Lemma 4.11.** *There exists a resolvable  $(6K_{36}, K_{1,3})$ -design.*

*Proof.* Start with a resolvable resolvable  $S(2, 3, 9)$ . Give weight 4 to all points and replace each block of a given resolution class with a copy of a resolvable  $K_{1,3}$ -RGDD of type  $4^3$  and index 6, which exists by Lemma 4.6. Finally, fill each group of size 4 with a copy of a resolvable  $(6K_4, K_{1,3})$ -design, which exists by Lemma 4.4.  $\square$

**Lemma 4.12.** *There exists a resolvable  $(5K_4, K_4 - e)$ -design.*

*Proof.* Let  $V = \{0, 1, 2, 3\}$  be the vertex set and consider the classes listed below:  $\{(1, 2, 0; 3)\}$ ,  $\{(3, 0, 2; 1)\}$ ,  $\{(1, 3, 0; 2)\}$ ,  $\{(2, 0, 1; 3)\}$ ,  $\{(1, 0, 2; 3)\}$ ,  $\{(2, 3, 1; 0)\}$ .  $\square$

**Lemma 4.13.** *There exists a resolvable  $(5K_8, K_4 - e)$ -design.*

*Proof.* Let  $V = Z_7 \cup \{\infty\}$  be the vertex set. The desired design is obtained by developing in  $Z_7$  the following base blocks:  
 $\{(\infty, 4, 5; 0), (1, 2, 3; 6)\}$ ,  $\{(0, 3, \infty; 2), (1, 6, 5; 4)\}$ .  $\square$

**Lemma 4.14.** *There exists a resolvable  $(5K_{12}, K_4 - e)$ -design.*

*Proof.* Let  $Z_8 \cup \{\infty_1, \infty_2, \infty_3, \infty_4\}$  be the vertex set. The desired design is obtained by filling the sets  $2Z_8 + i$ ,  $i = 0, 1$ , and  $\{\infty_1, \infty_2, \infty_3, \infty_4\}$  with a copy of a resolvable  $(5K_4, K_4 - e)$ -design (giving six parallel classes) and developing the two sets of base blocks in  $Z_8$   $\{(\infty_1, 1, 0; 2), (\infty_2, 4, 3; 5), (6, 7, \infty_3; \infty_4)\}$  and  $\{(\infty_3, 3, 0; 6), (\infty_4, 4, 1; 7), (2, 5, \infty_1; \infty_2)\}$ , which give the remaining parallel classes.  $\square$

**Lemma 4.15.** *There exists a resolvable  $(5K_{20}, K_4 - e)$ -design.*

*Proof.* Let  $V = Z_{19} \cup \{\infty\}$  be the vertex set. The desired design is obtained by developing in  $Z_{19}$  the following base blocks:  
 $\{(3, 4, 15; \infty), (1, 18, 9; 14), (2, 0, 5; 8), (6, 10, 12; 13), (7, 16, 11; 17)\}$ ,  
 $\{(0, \infty, 1; 15), (8, 10, 18; 5), (2, 16, 11; 9), (14, 13, 17; 7), (4, 6, 12; 3)\}$ .  $\square$

**Lemma 4.16.** *There exists a resolvable  $(5K_{24}, K_4 - e)$ -design.*

*Proof.* Start with a resolvable  $(K_4 - e)$ -RGDD of type  $4^6$  of index 5 (see [5]) and fill each group of size 4 with a copy of a resolvable  $(5K_4, K_4 - e)$ -design, which exists by Lemma 4.12.  $\square$

**Lemma 4.17.** *There exists an incomplete resolvable  $(K_4 - e)$ -design of order 28 and index  $\lambda = 5$  with a hole of size 8.*

*Proof.* Let the vertex set be  $Z_{20} \cup \{\infty_1, \infty_2, \dots, \infty_8\}$ . The desired design is obtained by filling each set  $5Z_{20} + i$ ,  $i = 0, 1, 2, 3, 4$ , with a copy of a resolvable  $(5K_4, K_4 - e)$ -design and developing the base blocks  $(0, 3, 1; 2)$ ,  $(0, 7, 6; 9)$ , each of which gives four partial parallel classes, and the following base blocks (partitioned into full parallel classes):

$\{(2, 3, \infty_1; \infty_2), (5, 11, \infty_3; \infty_4), (\infty_5, 9, 10; 12), (\infty_6, 1, 18; 19), (\infty_7, 6, 15; 17), (\infty_8, 7, 13; 14), (0, 8, 4; 16)\};$   
 $\{(10, 3, \infty_5; \infty_6), (5, 11, \infty_7; \infty_8), (\infty_1, 9, 2; 12), (\infty_2, 1, 18; 19), (\infty_3, 6, 15; 17), (\infty_4, 7, 13; 14), (0, 16, 4; 8)\}.$   $\square$

**Lemma 4.18.** *There exists a resolvable  $(5K_{36}, K_4 - e)$ -design.*

*Proof.* Take a resolvable  $(K_{36}, K_4 - e)$ -design (see [5]) and repeat the classes 5 times.  $\square$

**Lemma 4.19.** *There exists a resolvable  $(5K_{44}, K_4 - e)$ -design.*

*Proof.* It is sufficient to paste five copies of the maximum resolvable  $(K_4 - e)$ -packing of order 44 in [13], with leaves such that their edges can be suitably arranged so to give a new parallel class of copies of  $K_4 - e$ .  $\square$

**Lemma 4.20.** *There exists a resolvable  $(5K_{68}, K_4 - e)$ -design.*

*Proof.* Consider the minimum resolvable  $(K_4 - e)$ -covering of order 68 in [12], whose excess is contained into one parallel class so that removing its edges gives a maximum resolvable  $(K_4 - e)$ -packing of order 68 with leave a parallel class of kites. Now, paste five copies of the above packing with leaves such that their edges can be suitably arranged into four new parallel classes of copies of  $K_4 - e$ .  $\square$

## 5 The case $G = C_4, K_3 + e$

**Lemma 5.1.** *For every  $v \equiv 0 \pmod{4}$  there exists a resolvable  $(2K_v, C_4)$ -design.*



*Proof.* Let  $v = 4t$ . Start from a 2-RGDD  $\mathcal{G}$  of type  $2^t$  ([2]). Give weight 2 to every point of  $\mathcal{G}$  and for each block of a given resolution class of  $\mathcal{G}$  place 2 copies of a  $C_4$ -decomposition of type  $2^2$  ([4]). Finally, fill each group of size 4 with a copy of a resolvable  $(2K_4, C_4)$ -design which exists by Lemma 4.1.  $\square$

**Lemma 5.2.** *For every  $v \equiv 0 \pmod{4}$  there exists a resolvable  $(2K_v, K_3 + e)$ -design.*

*Proof.* Let  $v = 4t$ . The cases  $t = 1$  and  $t = 2$  correspond to a  $(2K_4, K_3 + e)$ -design and a  $(2K_8, K_3 + e)$ -design which exist by Lemma 4.2 and Lemma 4.3. For  $t > 2$ , start from a  $(K_3 + e)$ -RGDD  $\mathcal{G}$  of type  $4^t$  ([14]). Repeat each block of a given resolution class of  $\mathcal{G}$  twice and fill each group of size 4 with a copy of a resolvable  $(2K_4, K_3 + e)$ -design which exists by Lemma 4.2.  $\square$

## 6 The case $G = K_{1,3}$

**Lemma 6.1.** *For every  $v \equiv 4 \pmod{12}$ , there exists a resolvable  $(2K_v, K_{1,3})$ -design.*

*Proof.* Start from a resolvable  $(K_v, K_4)$ -design ([7]) and replace each block of a given resolution class with 2 copies of a resolvable  $(2K_4, K_{1,3})$ -design which exists by Lemma 4.4. This completes the proof.  $\square$

**Lemma 6.2.** *For every  $v \equiv 0, 8 \pmod{12}$  there exists a resolvable  $(6K_v, K_{1,3})$ -design.*

*Proof.* The cases  $v = 8, 12, 20, 24, 36$  are covered by Lemmas 4.7, 4.8, 4.9, 4.10, 4.11. We distinguish the following cases.

Case 1 :  $v \equiv 0 \pmod{12}$ ,  $v \geq 48$ .

Let  $v = 12t$ . Start from a 4-RGDD  $\mathcal{D}$  of type  $12^t$  ([2]) and replace each block of a given resolution class of  $\mathcal{D}$  with 3 copies of a resolvable  $(2K_4, K_{1,3})$ -design which exists by Lemma 4.4. Finally, fill each group of size 12 with a copy of a resolvable  $(6K_{12}, K_{1,3})$ -design which exists by Lemma 4.8.

Case 2 :  $v \equiv 8 \pmod{24}$ ,  $v \geq 32$ .

Let  $v = 8 + 24t$ . Start from a 4-RGDD  $\mathcal{D}$  of type  $8^{1+3t}$  ([2]) and replace each block of a given resolution class of  $\mathcal{D}$  with 3 copies of a resolvable  $(2K_4, K_{1,3})$ -design which exists by Lemma 4.4. Finally, fill each group of size 12 with a copy of a resolvable  $(6K_8, K_{1,3})$ -design which exists by Lemma 4.7.

Case 3:  $v \equiv 20 \pmod{24}$ ,  $v \geq 44$ .

Let  $v = 20 + 24t$ . Start from a 2-frame  $\mathcal{F}$  of type  $2^{2+3t}$ ,  $t \geq 1$ , with groups  $G_i$ ,  $i = 1, 2, \dots, 2 + 3t$ , ([2]); let  $X = \cup_{i=1}^{2+3t} G_i$ . Expand each point of  $X$  4 times and add a set  $H = \{\infty_1, \infty_2, \infty_3, \infty_4\}$ . For each  $x \in X$ , place on  $\{x\} \times Z_4$  3 copies of a resolvable  $(2K_4, K_{1,3})$ -design which exists by Lemma 4.4. This gives a set  $P$  of 12 partial parallel classes on  $X \times Z_4$ . Fill the hole  $H$  with 3 copies of a resolvable  $(2K_4, K_{1,3})$ -design which exists by Lemma 4.4. This gives a set  $P_H$  of 12 parallel classes on  $H$ . Combine  $P$  and  $P_H$  to obtain 12 full parallel classes of 3-stars. For  $j = 1, 2$ , let  $p_{i,j}$  be the 2 partial parallel classes which miss the group  $G_i$  and for each  $b \in p_{i,j}$ , place on  $b \times Z_4$  a copy of a resolvable  $K_{1,3}$ -RGDD of type  $4^2$  and index 6, which exists by Lemma 4.5. This gives a set  $P_{i,j}$  of 16 partial classes of 3-stars on  $X \setminus G_i$ . For each  $i = 1, 2, \dots, 2 + 3t$  place on  $H \cup (G_i \times Z_4)$  a copy of a resolvable  $K_{1,3}$ -RGDD of type  $4^3$  and index 6, which exists by Lemma 4.6; this gives a set  $P_i$  of 32 classes of 3-stars. Finally, combine the 32 classes of  $P_i$  with the classes of  $P_{i,1}$  and  $P_{i,2}$  to obtain the desired result.

□

## 7 The case $G = K_4 - e$

**Lemma 7.1.** *For every  $v \equiv 0, 4, 8, 12, 16 \pmod{24}$ , there exists a resolvable  $(5K_v, K_4 - e)$ -design.*

*Proof.* The cases  $v = 4, 8, 12, 24, 36$  are covered by Lemmas 4.12, 4.13, 4.14, 4.16, 4.18 and 4.15. We distinguish the following cases.

Case 1 :  $v \equiv 4 \pmod{12}$ .

Start from a resolvable  $(K_v, K_4)$ -design ([7]) and replace each block of every resolution class with a copy of a resolvable  $(5K_4, K_4 - e)$ -design which exists by Lemma 4.12.

Case 2 :  $v \equiv 0 \pmod{12}$ ,  $v \geq 48$ .

Let  $v = 12t$ . Start from a 4-RGDD  $\mathcal{D}$  of type  $12^t$  ([2]). Replace each block of every resolution class of  $\mathcal{D}$  with a copy of a resolvable  $(5K_4, K_4 - e)$ -design which exists by Lemma 4.12 and fill each group of size 12 with a copy of a resolvable  $(5K_{12}, K_4 - e)$ -design which exists by Lemma 4.14.

Case 3 :  $v \equiv 8 \pmod{24}$ ,  $v \geq 32$ .

Let  $v = 8 + 24t$ . Start from a 4-RGDD  $\mathcal{D}$  of type  $8^{1+3t}$  ([2]). Replace each block of every resolution class of  $\mathcal{D}$  with a copy of a resolvable  $(5K_4, K_4 - e)$ -design which exists by Lemma 4.12 and fill each group of size 8 with a copy of a resolvable  $(5K_8, K_4 - e)$ -design which exists by Lemma 4.13. This completes the proof.  $\square$

**Lemma 7.2.** *For every  $v \equiv 20 \pmod{120}$ , there exists a resolvable  $(5K_v, K_4 - e)$ -design.*

*Proof.* Let  $v = 20 + 120t$ . The case  $t = 0$  corresponds to a  $(5K_{20}, K_4 - e)$ -design which exists by Lemma 4.15. For  $t > 1$  take a 4-RGDD  $\mathcal{G}$  of type  $4^{1+6t}$  ([2]). Expand each point 5 times and replace each block of the resolution classes of  $\mathcal{G}$  with a copy of a resolvable  $(K_4 - e)$ -RGDD of type  $5^4$  and index 5 which exists by Lemma 3.2 of [13]. Finally, fill each group of size 20 with a copy of a resolvable  $(5K_{20}, K_4 - e)$ -design which exists by Lemma 4.15.  $\square$

**Lemma 7.3.** *For every  $v \equiv 44 \pmod{120}$ ,  $v > 44$ , there exists a resolvable  $(5K_v, K_4 - e)$ -design.*

*Proof.* Let  $v = 44 + 120t$ . The case  $t = 0$  corresponds to a  $(5K_{44}, K_4 - e)$ -design which exists by Lemma 4.19. For  $t > 0$  take a  $(K_4 - e)$ -RGDD  $\mathcal{G}$  of type  $4^{1+5(2+6t)}$  and index 5 ([13]) and fill each group of size 4 with a copy of a resolvable  $(5K_4, K_4 - e)$ -design which exists by Lemma 4.12.  $\square$

**Lemma 7.4.** *For every  $v \equiv 68 \pmod{120}$ , there exists a resolvable  $(5K_v, K_4 - e)$ -design.*

*Proof.* Let  $v = 68 + 120t$ . The case  $t = 0$  corresponds to a  $(5K_{68}, K_4 - e)$ -design which exists by Lemma 4.20. For  $t > 0$ , let  $\mathcal{F}$  be a  $(K_4 - e)$ -frame of type  $20^{3+6t}$  ([13]) with groups  $G_i$ ,  $i = 1, 2, \dots, 3 + 6t$ . Add a set  $H = \{\infty_1, \infty_2, \dots, \infty_8\}$ . For each  $i = 1, 2, \dots, 3 + 6t$ , let  $P_i$  the set of the partial parallel classes which miss the group  $G_i$ , taken 5 times. Place on  $G_i \cup H$  a copy of an incomplete resolvable  $(K_4 - e)$ -design of order 28 and index 5, having the set of 8 infinite points as hole, and combine its full classes with the partial classes of  $P_i$  so to obtain  $40(3 + 6t)$  parallel classes on  $H \cup (\cup_{i=1}^{3+6t} G_i)$ . Fill the hole  $H$  with a copy of a resolvable  $(5K_8, K_4 - e)$ -design which exists by Lemma 4.13 and combine its 14 classes with the partial classes of  $P_i$  so to obtain 14 parallel classes. The result is a resolvable  $(5K_v, K_4 - e)$ -design.  $\square$

**Lemma 7.5.** *For every  $v \equiv 92 \pmod{120}$ , there exists a resolvable  $(5K_v, K_4 - e)$ -design.*

*Proof.* Let  $v = 92 + 120t$ . Let  $P_1 = \{\{i, 1+i\}, i \in Z_v\}$  and  $P_{\frac{v}{2}-1} = \{\{i, \frac{v}{2}-1+i\}, i \in Z_v\}$  be two sets of  $v$  edges of  $K_v$  and  $P_{\frac{v}{2}} = \{\{i, \frac{v}{2}+i\}, i = 0, 1, \dots, \frac{v}{2}-1\}$  be a set of  $\frac{v}{2}$  edges of  $K_v$ . The edges of  $P_1 \cup P_{\frac{v}{2}-1} \cup P_{\frac{v}{2}}$  can be decomposed into a set of 5 1-factors  $F_j, j = 1, 2, 3, 4, 5$  ([11]). For each  $j = 1, 2, 3, 4, 5$  take on  $K_v$  a  $(K_4 - e)$ -RGDD  $\mathcal{G}$  of type  $2^{6+10(4+6t)}$  ([13]) with the edges of  $F_j$  as groups. Finally, the missing edges of  $P_1 \cup P_{\frac{v}{2}-1} \cup P_{\frac{v}{2}}$  (until now arranged only 4 times) can be decomposed into the set  $\{(i, \frac{v}{2}+i, \frac{v}{2}+1+i; 1+i), i = 0, 1, \dots, \frac{v}{2}-1\}$  of  $\frac{v}{2}$  copies of  $K_4 - e$ , which generate the remaining two parallel classes (the first class is obtained for  $i = 0, 2, \dots, \frac{v}{2}-2$ , while the second class for  $i = 1, 3, \dots, \frac{v}{2}-1$ ).  $\square$

**Lemma 7.6.** *For every  $v \equiv 116 \pmod{120}$ , there exists a resolvable  $(K_v, K_4 - e)$ -design.*

*Proof.* Since the condition  $v \equiv 116 \pmod{120}$  implies  $v \equiv 16 \pmod{20}$ , the proof follows by [5, 15].  $\square$

## 8 Conclusion

We are now in a position to prove the following main result.

**Theorem 8.1.** *The necessary conditions of Lemma 1.1 are sufficient for the existence of a resolvable  $(\lambda K_v, G)$ -design in the cases when  $G = C_4, K_3 + e, K_{1,3}, K_4 - e$ .*

*Proof.* Let  $G \in \{C_4, K_3 + e\}$ . For any  $v \equiv 0 \pmod{4}$  and  $\lambda = 2\mu$  ( $\mu \geq 1$ ), combine  $\mu$  copies of a resolvable  $(2K_v, G)$ -design, which exists by Lemmas 5.1 and 5.2.

Let  $G = K_{1,3}$ . For any  $v \equiv 4 \pmod{12}$  and  $\lambda = 2\mu$  ( $\mu \geq 1$ ), combine  $\mu$  copies of a resolvable  $(2K_v, K_{1,3})$ -design, which exists by Lemma 6.1; while, for any  $v \equiv 0, 8 \pmod{12}$  and  $\lambda = 6\mu$  ( $\mu \geq 1$ ), combine  $\mu$  copies of a resolvable  $(6K_v, K_{1,3})$ -design, which exists by Lemma 6.2.

Let  $G = K_4 - e$ . For any  $v \equiv 0, 1, 8, 12 \pmod{20}$  and  $\lambda = 5\mu$  ( $\mu \geq 1$ ), combine  $\mu$  copies of a resolvable  $(5K_v, K_4 - e)$ -design, which exists by Lemmas 7.1 -7.4; while, for any  $v \equiv 16 \pmod{20}$  combine  $\lambda$  copies of a resolvable  $(K_v, K_4 - e)$ -design, which exists by Theorem 3 of [15].  $\square$

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